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LETTER TO THE EDITOR

A special case of Neumann's system and the Kowalewski–Chaplygin–Goryachev top

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Abstract. We present L operators both for the special case of Neumann's system and the Kowalewski-Chaplygin-Goryachev top. These are quantum systems integrable by the quantum inverse scattering (R-matrix) method. L operators, being 2×2 matrices, satisfy an algebra generated by an R matrix of the XXX type. The close connection between the two models is demonstrated. We carry out a non-obvious separation of variables and also give a dynamical group scheme for the eigenstate problem of Neumann's system. This separation differs from those in Euler angles and allows us to find eigenenergies in an effective way.

The quantum inverse scattering method (QISM) worked out by Faddeev and his collaborators provides a general scheme for studying integrable models of statistical physics, quantum field theory and finite-dimensional quantum mechanics [1, 2]. To study eigenstates of constants of motion in QISM the algebraic Bethe ansatz is usually applied, but there are such integrable systems for which the algebraic Bethe ansatz does not work due to the non-existence of an invariant vacuum state. For such cases the special procedure of separation of variables in QISM was introduced [3–5]. It was called a 'functional Bethe ansatz'.

The main object of QISM is an associative algebra defined by the generators $T_{\alpha\beta}(u)$ $(\alpha, \beta = 1, \ldots, d;\ u \in \mathbb{C})$ considered as the elements of the square matrix $T(u)$ with the commutational relation

$$R(u-v)\overset{1}{T}(u)\overset{2}{T}(v) = \overset{2}{T}(v)\overset{1}{T}(u)R(u-v) \tag{1}$$

where

$$\overset{1}{T}(u) = T(u)\otimes I_d \qquad \overset{2}{T}(v) = I_d\otimes T(v).$$

The matrix $R(u)\in \mathrm{Aut}(\mathbb{C}^d\otimes\mathbb{C}^d)$ is a solution of the quantum Yang–Baxter equation [1]. The constants of motion are extracted from the trace of the matrix $T(u)$.

Let us consider the simplest case, $d=2$, when $T(u)$ is the 2×2 matrix

$$T(u) = \begin{pmatrix} A & B \\ C & D \end{pmatrix}(u) \qquad \tau(u) = A(u) + D(u) \tag{2}$$

and $R(u)$ is the R matrix of the XXX type as follows:

$$R(u)=u+\mathrm{i}\varkappa P \qquad P=\begin{pmatrix}1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&1\end{pmatrix} \qquad \varkappa\in\mathbb{C}. \tag{3}$$

In this letter a new representation of algebra $T(u)$ (L operator) is constructed for a given R matrix (3). This new L operator is connected to the integrable model which is a special case of Neumann's system in quantum mechanics. The Hamiltonian has the form

$$H=\tfrac{1}{2}(J_1^2+J_2^2-b^2x_3^2)+\alpha/x_3^2. \tag{4}$$

Variables J_i, x_i, $i=1,2,3$, are generators of the Lie algebra e(3) obeying the following commutation relations:

$$[J_i,J_j]=-\mathrm{i}\varepsilon_{ijk}J_k \qquad [J_i,x_j]=-\mathrm{i}\varepsilon_{ijk}x_k \qquad [x_i,x_j]=0. \tag{5}$$

Further, we restrict the values of the Casimir operators by

$$a^2=\sum_{i=1}^{3}x_1^2=1 \qquad l=\sum_{i=1}^{3}x_iJ_i=0. \tag{6}$$

The classical model similar to (4) was studied, for instance, in [6]. Having the L operator we carry out separation of variables [3-5] and also give the dynamical group $G=\mathrm{SO}(2.1)\times\mathrm{SO}(2.1)$ for this system. The separation proposed allows us to numerically compute the spectrum of energy more effectively than using the usual procedure of eigenfunction factorisation in Euler angles. Indeed, in our approach we obtain both the separation of variables or eigenfunction factorisation and also the suitable basis for eigenfunctions of the separated one-dimensional spectral problems.

Recently Sklyanin [7] described a new class of boundary conditions for quantum systems integrable by means of QISM. Two algebras $\mathcal{T}_\pm(u)$ are analogues of the algebra $T(u)$ in (1)

$$R(u-v)\overset{1}{\mathcal{T}}_-(u)R(u+v-\mathrm{i}\varkappa)\overset{2}{\mathcal{T}}_-(v)=\overset{2}{\mathcal{T}}_-(v)R(u+v-\mathrm{i}\varkappa)\overset{1}{\mathcal{T}}_-(u)R(u-v) \tag{7a}$$

$$R(-u+v)\overset{1}{\mathcal{T}}{}_+^{t_1}(u)R(-u-v-\mathrm{i}\varkappa)\overset{2}{\mathcal{T}}{}_+^{t_2}(v)=\overset{2}{\mathcal{T}}{}_+^{t_2}(v)R(-u-v-\mathrm{i}\varkappa)\overset{1}{\mathcal{T}}{}_+^{t_1}(u)R(-u+v) \tag{7b}$$

where t_1 and t_2 are the matrix transpositions in the first and second spaces, respectively. The quantities $\tau(u)=\mathrm{Tr}\,\mathcal{T}_+(u)\mathcal{T}_-(u)$ defined in the direct product $\mathcal{T}_+\times\mathcal{T}_-$ form a commutative family $[\tau(u_1),\tau(u_2)]=0$ for any u_1, u_2 (theorem 1 in [7]). Hence $\tau(u)$ is the generating function of the constants of motion.

Representations of the algebras $\mathcal{T}_\pm(u)$ can be constructed having that of the algebra $T(u)$ (1) and that of the algebras $\mathcal{T}_\pm(u)$ in $\mathbb{C}^1$, i.e. c-number matrices [7]. In this manner, we connect the special case of Neumann's system with another integrable model which we call the Kowalewski-Chaplygin-Goryachev top (KCGT). The Hamiltonian of KCGT is

$$H=\tfrac{1}{2}(J_1^2+J_2^2+2J_3^2)+c_1x_1+c_2x_2+c_3(x_1^2-x_2^2)+c_4x_1x_2+c_5/x_3^2 \tag{8}$$

where c_i, $i=1,\ldots,5$, are arbitrary constants. Notice that the system is integrable provided that $l=\Sigma_{i=1}^{3}x_iJ_i=0$. In the case $c_3=c_4=c_5=0$ KCGT turns into Kowalewski's top. In the classical mechanics the integrability of the dynamical Euler equations for

KCGT was established by Chaplygin for special values $c_4 = c_5 = 0$ in 1903 [8] and in the general case by Goryachev in 1916 [9]. In this letter we find representations of the algebras $\mathcal{T}_\pm(u)$ for KCGT.

Let us consider the following ansatz for the L operator:

$$L(u) = \begin{pmatrix} A & B \\ C & D \end{pmatrix}(u) = \begin{pmatrix} y_0 u^2 + y_2 u + y_1 & y_4^+ u + y_6^+ \\ y_4^- u + y_6^- & y_3 \end{pmatrix} \tag{9}$$

where $y_4^\pm = y_4 \pm \mathrm{i} y_5$, $y_6^\pm = y_6 \pm \mathrm{i} y_7$ and $y_0, \ldots, y_7$ are eight not yet defined affine variables. Suppose that the L operator satisfies equation (1) of the R-matrix quadratic algebra. It generates some quadratic algebra $\mathrm{A}^{(2)}$ for y_n analogous to $\mathcal{F}$ in [10]. The algebra $\mathrm{A}^{(2)}$ has the centre

$$\begin{aligned} &Q_0 = y_0 \\ &Q_1 = y_0 y_3 - y_4^2 - y_5^2 \\ &Q_2 = y_2 y_3 - 2y_4 y_6 - 2y_5 y_7 \\ &Q_3 = y_6^2 + y_7^2 - \tfrac{1}{2}\{y_1, y_3\} + \tfrac{1}{2}\varkappa^2 y_0 y_3 \\ &\{y_i, y_k\} = y_i y_k + y_k y_i. \end{aligned} \tag{10}$$

The quantum determinant [2] of the L operator is the generating function of the centre elements and has the form

$$\begin{aligned} d(u) &= A(u + \tfrac{1}{2}\mathrm{i}\varkappa) D(u - \tfrac{1}{2}\mathrm{i}\varkappa) - B(u + \tfrac{1}{2}\mathrm{i}\varkappa) C(u - \tfrac{1}{2}\mathrm{i}\varkappa) \\ &= Q_1 u^2 + Q_2 u - Q_3 + \tfrac{1}{4}\varkappa^2 Q_1. \end{aligned} \tag{11}$$

So, as y_0 belongs to the centre, we can see that, in the limit $y_0 \to 0$, the quadratic algebra $\mathrm{A}^{(2)}$ contracts to $\mathrm{sl}(2, \mathbb{C})$. Moreover, its real form is fixed by the conditions $\bar{\varkappa} = -\varkappa$, $(xy)^* = y^* x^*$ for any $x, y \in \mathrm{A}^{(2)}$ and, under the additional restriction $Q_2 = 0$, turns into the Lie algebra so(2.1).

There are two natural realisations of the quadratic algebra $\mathrm{A}^{(2)}$ by generators of the Lie algebras $\mathrm{p}(1.1) \oplus \mathrm{p}(1.1)$ and e(3). The algebra $\mathrm{p}(1.1)^{\oplus 2}$ with generators p_n, $e_n^\pm$, $n = 1, 2$, obeying commutation relations

$$[p_n, e_k^\pm] = \pm \mathrm{i}\varkappa e_k^\pm \delta_{nk} \qquad [p_n, p_k] = [e_n^\pm, e_k^\pm] = 0 \qquad e_n^+ e_n^- = 1 \tag{12}$$

is connected to $\mathrm{A}^{(2)}$ by the following equations:

$$\begin{aligned} &y_0 = 1 && y_4^+ = -e_1^+ \\ &y_1 = p_1 p_2 - e_2^+ e_1^- && y_4^- = e_2^- \\ &y_2 = -(p_1 + p_2) && y_6^+ = p_2 e_1^+ \\ &y_3 = -e_2^- e_1^+ && y_6^- = -p_1 e_2^-. \end{aligned} \tag{13}$$

The centre elements (10) have the values

$$Q_0 = 1 \qquad Q_1 = Q_2 = 0 \qquad Q_3 = -1. \tag{14}$$

Generators J_i, x_i (5) of the Lie algebra e(3), provided that $l = \Sigma_{i=1}^3 x_i J_i = 0$, define that of $\mathrm{A}^{(2)}$ as follows ($\varkappa = 2\mathrm{i}$):

$$\begin{aligned} &y_0 = 1 && y_4 = \mathrm{i} b x_1 \\ &y_1 = -(J_1^2 + J_2^2 + \tfrac{1}{4} + 2\alpha / x_3^2) && y_5 = \mathrm{i} b x_2 \\ &y_2 = -2J_3 && y_6 = -\tfrac{1}{2}\mathrm{i} b \{x_3, J_1\} \\ &y_3 = b^2 x_3^2 && y_7 = -\tfrac{1}{2}\mathrm{i} b \{x_3, J_2\} \end{aligned} \tag{15}$$

where α, b are arbitrary constants. The centre elements (10) now take the values

$$Q_0=1 \qquad Q_1=b^2 \qquad Q_2=0 \qquad Q_3=b^2(2\alpha-\tfrac{3}{4}). \tag{16}$$

Affine variables y_n were originally used by Bechlivanidis and van Moerbeke [11] without any discussion of their algebraic properties.

In terms of p(1.1)⊕p(1.1) algebra (13) the L operator (9) coincides with the monodromy matrix of the quantum periodic Toda lattice with two particles [4] and will not be considered further. In the e(3) realisation (15) the L operator acquires the form

$$L(u)=\begin{pmatrix} A & B \\ C & D \end{pmatrix}=\begin{pmatrix} u^2-2J_3u-J_1^2-J_2^2-\frac{1}{4}-2\alpha/x_3^2 & \mathrm{i}b(x_+u-\frac{1}{2}\{x_3,J_+\}) \\ \mathrm{i}b(x_-u-\frac{1}{2}\{x_3,J_-\}) & b^2x_3^2 \end{pmatrix} \tag{17}$$

where $x_\pm=x_1\pm\mathrm{i}x_2$, $J_\pm=J_1\pm\mathrm{i}J_2$. The trace $\tau(u)$ of $L(u)$ is the generating polynomial of the constants of motion

$$\begin{aligned} &\tau(u)=u^2-2Gu-2(H+\tfrac{1}{8})\\ &H=\tfrac{1}{2}(J_1^2+J_2^2-b^2x_3^2)+\alpha/x_3^2\\ &G=J_3. \end{aligned} \tag{18}$$

Thus $L(u)$ corresponds to the special case of Neumann's system.

To find eigenstates of $\tau(u)$ in (18) we apply the method of separation of variables worked out in [3, 4] and developed in [5]. Separated variables are determined as commuting roots of the operator equation $C(u)=0$. This equation for the L operator (17) has one nilpotent root, in contrast to the Goryachev-Chaplygin top and the Toda lattice [3-5]. To overcome this difficulty we transform the L operator (17) into the $\tilde{L}$ operator

$$\tilde{L}(u)=\frac{\sigma_1+\sigma_3}{\sqrt{2}}L(u)\frac{\sigma_1+\sigma_3}{\sqrt{2}}=\begin{pmatrix} \tilde{A} & \tilde{B} \\ \tilde{C} & \tilde{D} \end{pmatrix}(u) \tag{19}$$

where σ_i are Pauli matrices. $\tilde{L}(u)$ satisfies equation (1) with the same R matrix (3), the trace of $\tilde{L}(u)$ coincides with that of $L(u)$, and therefore $\tilde{L}(u)$ describes just the same dynamical system. The quantum determinant $d(u)$ of $\tilde{L}(u)$ is

$$d(u)=b^2(u^2-\tfrac{1}{4}-2\alpha). \tag{20}$$

Commuting separated variables are now defined as roots of the square equation

$$\tilde{C}(u)=0 \tag{21}$$

and are given by

$$u_{1,2}=J_3+bx_2\pm[(J_3+bx_2)^2+J_1^2+\tfrac{1}{4}+2\alpha/x_3^2+(J_2-bx_3)^2]^{1/2}. \tag{22}$$

Variables $u_{1,2}$ are Hermitian operators provided that $\alpha\geqslant-\frac{1}{8}$; otherwise the operator under the square root is not positively defined.

Let us introduce two operators $m_j^\mp$ for each u_j using Lagrange interpolation of the operator polynomials $\tilde{A}(u)$ and $\tilde{D}(u)$ as follows:

$$\begin{aligned}\tilde{A}(u)=\tfrac{1}{2}(u-u_1)(u-u_2)&+\frac{u_1-u}{(u_1-u_2)^{1/2}}m_2^-\frac{1}{(u_1-u_2)^{1/2}}\\ &+\frac{u-u_2}{(u_1-u_2)^{1/2}}m_1^-\frac{1}{(u_1-u_2)^{1/2}}\end{aligned} \tag{23a}$$

$$\tilde{D}(u)=\tfrac{1}{2}(u-u_1)(u-u_2)+\frac{u_1-u}{(u_1-u_2)^{1/2}}\,m_2^+\,\frac{1}{(u_1-u_2)^{1/2}}$$

$$+\frac{u-u_2}{(u_1-u_2)^{1/2}}\,m_1^+\,\frac{1}{(u_1-u_2)^{1/2}}. \qquad (23b)$$

Explicit formulae for $m_j^{\pm}$ are

$$m_j^- = \frac{1}{(u_1-u_2)^{1/2}}\,\tilde{A}(u \overset{\frown}{=} u_j)(u_1-u_2)^{1/2}$$

$$m_j^+ = \frac{1}{(u_1-u_2)^{1/2}}\,\tilde{D}(u \overset{\frown}{=} u_j)(u_1-u_2)^{1/2} \qquad (24)$$

where the left substitution of $u_{1,2}$ in $\tilde{A}(u)$ and $\tilde{D}(u)$ is chosen. Notice that the difference u_1-u_2 is positively defined. Operators u_j, $m_j^{\pm}$ possess the following properties:

$$(m_j^{\pm})^* = m_j^{\mp} \qquad u_j^* = u_j \qquad (25)$$

$$[m_j^{\pm}, u_k] = \pm 2\delta_{jk} m_k^{\pm} \qquad (26a)$$

$$[m_j^{\pm}, m_k^{\pm}] = [u_j, u_k] = 0 \qquad (26b)$$

$$m_j^- m_j^+ = d(u_j-1) \qquad m_j^+ m_j^- = d(u_j+1) \qquad (26c)$$

where $d(u)$ is the quantum determinant (20). Equations (25) and (26) are derived from the fundamental relation (1) by means of the technique developed in [3, 4]. The rewriting of the L operator in terms of u_j, $m_j^{\pm}$ is single-valued.

Notice that the operators

$$Z_n^{(1)} = \frac{1}{4b}(m_n^+ + m_n^-)$$

$$Z_n^{(2)} = \frac{\mathrm{i}}{4b}(m_n^+ - m_n^-) \qquad (27)$$

$$Z_n^{(3)} = \tfrac{1}{2}u_n$$

obey the standard commutation relations of algebra $\mathrm{g} = \mathrm{so}(2.1)\oplus\mathrm{so}(2.1)$

$$[Z_m^{(\alpha)}, Z_n^{(\beta)}] = -\mathrm{i}\delta_{mn}\varepsilon_{\alpha\beta\gamma}\Delta_{\gamma\delta}Z_m^{(\delta)} \qquad \Delta = \mathrm{diag}(-1,-1,1) \qquad (28)$$

and the Casimir operators are

$$C_n = \Delta_{\gamma\beta} Z_n^{(\gamma)} Z_n^{(\beta)} = -\tfrac{3}{16} + \tfrac{1}{2}\alpha = j(j+1) \qquad (29)$$

where equations (20), (26c) and (27) were used. Eventually we have two Lie algebras so(2.1) with Hermitian generators (27). Consider discrete $D^{\pm}$ series of unitary irreducible representations of so(2.1). In accordance with (29) we are interested in the following 'spins':

$$j_{\pm} = -\tfrac{1}{2} \pm (\tfrac{1}{16} + \tfrac{1}{2}\alpha)^{1/2}. \qquad (30)$$

In order to stay in the $D^{\pm}$ series of irreducible representations of so(2.1) we have to restrict α by

$$-\tfrac{1}{8} \leqslant \alpha < \tfrac{3}{8}. \qquad (31)$$

Then choosing a representation in which compact generators $Z_n^{(3)}=\frac{1}{2}u_n$ are diagonal, 'spin' values $j_\pm$ (30) define the spectrum S of operators $u_{1,2}$ which consists of the two semi-infinite equidistant lattices $S=S_+\cup S_-$

$$S_\pm=\{(u_1,u_2)\in\mathbb{R}^2\colon (u_1,u_2)=(-2j_\pm+2n_1,2j_\pm-2n_2),\ n_1,n_2=0,1,2,\ldots\} \tag{32}$$

where we put D^+ for u_1 and D^- for u_2 in accordance with the inequality $u_1>u_2$ (22). The eigenfunction space of the system can be realised as the space $\mathscr{L}_2(S)$ of square-summable functions on the spectrum S:

$$\mathscr{L}_2(S)=\left\{f(u_1,u_2)\colon (u_1,u_2)\in S,\ \sum_{(u_1,u_2)\in S}|f(u_1,u_2)|^2<\infty\right\}. \tag{33}$$

It is easy to check that the operators $m_n^\pm$, acting on eigenfunctions φ of u_n,

$$(u_n\varphi)(u_1,u_2)=u_n\varphi(u_1,u_2)$$

as follows:

$$\begin{aligned}(m_1^\pm\varphi)(u_1,u_2)&=d^{1/2}(u_1\pm1)\varphi(u_1\pm2,u_2)\\(m_2^\pm\varphi)(u_1,u_2)&=d^{1/2}(u_2\pm1)\varphi(u_1,u_2\pm2)\end{aligned} \tag{34}$$

obey all the relations (25) and (26). Further, we consider the spectral problem for the generating function $\tau(u)=\tilde{A}(u)+\tilde{D}(u)=u^2-2Gu-2(H+\frac{1}{8})$. It has the form (see (23))

$$\begin{aligned}(t(u)f)(u_1,u_2)&\\&=(u-u_1)(u-u_2)f(u_1,u_2)\\&\quad+(u-u_2)(u_1-u_2)^{-1/2}(m_1^++m_1^-)(u_1-u_2)^{-1/2}f(u_1,u_2)\\&\quad+(u_1-u)(u_1-u_2)^{-1/2}(m_2^++m_2^-)(u_1-u_2)^{-1/2}f(u_1,u_2)\end{aligned} \tag{35}$$

where $f\in\mathscr{L}_2(S)$ is the eigenfunction of $\tau(u)$ and $t(u)=u^2-2mu-2(h+\frac{1}{8})$, where m, h are eigenvalues of G and H, respectively. It appears that, for the separation of variables u_1 and u_2, we ought to write

$$f(u_1,u_2)=(u_1-u_2)^{1/2}\varphi(u_1,u_2). \tag{36}$$

Now, considering $u=u_n$, $n=1,2$, in (35), we have two separated one-dimensional equations:

$$t(u_n)\varphi_n(u_n)=d^{1/2}(u_n-1)\varphi_n(u_n-2)+d^{1/2}(u_n+1)\varphi_n(u_n+2) \tag{37}$$

where the function $\varphi(u_1,u_2)$ is factorised:

$$\varphi(u_1,u_2)=\varphi_1(u_1)\varphi_2(u_2). \tag{38}$$

Such a form of the function $\varphi(u_1,u_2)$ allows us to say something about separation of variables and reflects the structure of the direct sum of the algebra (27) $\mathrm{g}=\mathrm{so}(2.1)\oplus\mathrm{so}(2.1)$.

Two one-dimensional spectral problems are the three-term recursion relations for the coefficients $\varphi_n(u_n)$, where the variables $u_{1,2}$ belong to the lattice S depending on the value of α. A complete numerical calculation of the eigenstates will be published elsewhere.

In conclusion we connect the special case of Neumann's system governed by the Hamiltonian (18) with KCGT (8) and construct the L operator of the latter. In our case the representations of the algebras $\mathcal{T}_\pm(u)$ are

$$\begin{aligned}\mathcal{T}_-(u) &= L(u)K_-(u-\tfrac{1}{2}\mathrm{i}\varkappa)\sigma_2 L^{\mathrm{T}}(-u)\sigma_2 \\ \mathcal{T}_+(u) &= K_+(u+\tfrac{1}{2}\mathrm{i}\varkappa)\end{aligned} \tag{39}$$

where $L(u)$ is the L operator (17) and the c-number matrices $K_\pm(u)$ have the form

$$K_-(u) = \begin{pmatrix} \alpha_1 & u \\ -\beta_1 u & \alpha_1 \end{pmatrix} \qquad K_+(u) = \begin{pmatrix} \alpha_2 & \beta_2 u \\ -u & \alpha_2 \end{pmatrix} \tag{40}$$

where α_i, β_i are arbitrary complex constants. Then the generating function $\tau(u) = \operatorname{Tr} \mathcal{T}_+(u)\mathcal{T}_-(u)$ of the constants of motion gives us an integrable system with the Hamiltonian of KCGT

$$H = \tfrac{1}{2}(J_1^2 + J_2^2 + 2J_3^2) + c_1 x_1 + c_2 x_2 + c_3(x_1^2 - x_2^2) + c_4 x_1 x_2 + c_5/x_3^2 \tag{41}$$

where $c_1 = \frac{1}{2}\mathrm{i}b(\alpha_2 - \alpha_1)$, $c_2 = \frac{1}{2}b(\alpha_1 + \alpha_2)$, $c_3 = -\frac{1}{4}b^2(\beta_1 + \beta_2)$, $c_4 = \frac{1}{2}b^2\mathrm{i}(\beta_2 - \beta_1)$, $c_5 = \alpha$. The separation of variables for KCGT is still unknown.

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Note added in proof. We must mention that the eigenfunctions of $\tau(u)$ in (18) in terms of Euler angles are connected to the special polyspheroidal harmonics.

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